7.1.2 What's the chance of rolling a six in *n* rolls of a dice?

Let's start with a simple question: *What's the chances of rolling a six in one roll of a dice?* Out of six sides of a dice, only one side gives you a six. Therefore, the chances of rolling a six with one roll of a dice are $1/6$. **Next question:** *What's the chances of rolling a six in two rolls of a dice?* As we learned from the first question, for one roll the chances are $1/6$. Out of six sides of a dice, five sides give a non-six. Thus, chances of not rolling a six with one roll are $5/6$. However, with two rolls we have to consider all possible permutations of compound probability of the two consecutive events of rolling a dice:

- 1^{st} roll is a six, 2^{nd} roll is a non-six
- 1^{st} roll is a non-six, 2^{nd} roll is a six
- 1^{st} roll is a six, 2^{nd} roll is a six
- 1^{st} roll is a non-six, 2^{nd} roll is a non-six

In mathematically terms: Given **A** is the set of all possible permutations of two consecutive rolls of a dice, where **S** denotes the roll of a six, **N** denotes the roll of a non-six, we thus can express it as $\mathbf{A} = \{\{S, N\}, \{N, S\}, \{S, S\}, \{N, N\}\}$ Let's look at each tuple.

For $\{S, N\}$, the compound probability **P** of rolling a six and a non-six is . . .

$$\mathbf{P} = \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$$

For $\{N, S\}$ the compound probability **P** of rolling a non-six and a six is . . .

$$\mathbf{P} = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$$

For $\{S, S\}$ the compound probability **P** of rolling two consecutive sixes is . . .

$$\mathbf{P} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

For $\{N, N\}$ the compound probability **P** of rolling two consecutive non-sixes is . . .

$$\mathbf{P} = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$$

Set $\mathbf{A} = \{\{S, N\}, \{N, S\}, \{S, S\}, \{N, N\}\}$ contains all permutations as tuples of *two consecutive events* of rolling one dice and getting either a six or a non-six. However, the question of *What's the chances of rolling a six in two rolls of a dice?* indicates that we don't necessarily need two consecutive events to produce the desired result: We have a chance to roll a six in one roll!

Amending set **A** to include single events, where **S** denotes the roll of a six, **N** denotes the roll of a non-six, we arrive at $\mathbf{A} = \{\{S\}, \{N\}, \{S, N\}, \{N, S\}, \{S, S\}, \{N, N\}\}$

Set $\mathbf{A}$ now contains all possible events of rolling a dice *one time* and *two consecutive times*, where **S** denotes the roll of a six, **N** denotes the roll of a non-six. However, set $\mathbf{A}$ contains events that do not produce a desired result. Let us create a sub-set of *undesired events* as set $\mathbf{B} = \{\{N\}, \{S, N\}, \{S, S\}, \{N, N\}\}$

As for event $\{N\}$, if we don't roll a six on the 1st roll we roll a 2nd time. Thus, event $\{N\}$ never actually happens because it becomes part of an event that consist of two consecutive rolls where the 1st roll was a non-six. As for the compound events $\{S, N\}$ and $\{S, S\}$, if we rolled a six on the 1st roll a 2nd roll never happens. As for event $\{N, N\}$, rolling two consecutive non-sixes is an undesired event.

If we subtract the set of *undesired events* $\mathbf{B}$ from the set of *all possible events* $\mathbf{A}$ we get the set of *desired events* $\mathbf{C} = \{\{S\}, \{N, S\}\}$

As for event $\{S\}$, if we roll a six on the 1st roll we don't roll a 2nd time. As for event $\{N, S\}$, if we don't roll a six on the 1st roll we roll a 2nd time but only consider it a desired result if it is a six.

Next question: *What's the chances of rolling a six in three rolls of a dice?* Since set $\mathbf{C}$ already incorporates *desired events* for one and two rolls, we can extend it to include the single desired event for three rolls, which is $\{N, N, S\}$ because if we make a six on the 1st or 2nd roll we don't roll a 3rd time. Thus, set $\mathbf{C} = \{\{S\}, \{N, S\}, \{N, N, S\}\}$

For $\{N, N, S\}$ the compound probability $\mathbf{P}$ of rolling two non-sixes and a six is ...

$$\mathbf{P} = \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216}$$

Next question: *What's the chances of rolling a six in four rolls of a dice?* As in the previous question, we only need to incorporate desired events into $\mathbf{C}$. The only *desired event* for four rolls is $\{N, N, N, S\}$. Thus, set $\mathbf{C} = \{\{S\}, \{N, S\}, \{N, N, S\}, \{N, N, N, S\}\}$

For $\{N, N, N, S\}$ the compound probability $\mathbf{P}$ of rolling three non-sixes and a six is ...

$$\mathbf{P} = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{125}{1296}$$

Towards an answer of the original question *(What's the chances of rolling a six in* $\mathbf{n}$ *rolls of a dice?)*: *Do you see the pattern?* We could go on and on and add more rolls but the pattern remains: Except for the desired event $\{S\}$ of the 1st roll, every other event reflects itself within $\mathbf{C}$ as a number of consecutive N's *before* S.

Given $\mathbf{n}$ the number of rolls ...

For $\mathbf{n} = 2 \ldots$

$\mathbf{C} = \{\{S\}, \{\text{N}, S\}\}$
$\mathbf{P} = {}^{1}/_{6} + ({}^{5}/_{6})^{\mathbf{1}} \times {}^{1}/_{6}$

For $\mathbf{n} = 3 \ldots$

$\mathbf{C} = \{\{S\}, \{\text{N}, S\}, \{\text{N}, \text{N}, S\}\}$
$\mathbf{P} = {}^{1}/_{6} + ({}^{5}/_{6})^{\mathbf{1}} \times {}^{1}/_{6} + ({}^{5}/_{6})^{\mathbf{2}} \times {}^{1}/_{6}$

For $\mathbf{n} = 4 \ldots$

$\mathbf{C} = \{\{S\}, \{\text{N}, S\}, \{\text{N}, \text{N}, S\}, \{\text{N}, \text{N}, \text{N}, S\}\}$
$\mathbf{P} = {}^{1}/_{6} + ({}^{5}/_{6})^{\mathbf{1}} \times {}^{1}/_{6} + ({}^{5}/_{6})^{\mathbf{2}} \times {}^{1}/_{6} + ({}^{5}/_{6})^{\mathbf{3}} \times {}^{1}/_{6}$

etc.

In order to *stress* the evolving pattern of $({}^{5}/_{6})^{\mathbf{i}}$ that shows in the probability of $\mathbf{P}$ for each value of $\mathbf{n}$, I wrote $({}^{5}/_{6})^{\mathbf{1}}$ instead of ${}^{5}/_{6}$. Because ${}^{5}/_{6} = ({}^{5}/_{6})^{\mathbf{1}}$, and written as such, we can relate $\mathbf{n}$ to the exponent $\mathbf{i}$ of $({}^{5}/_{6})^{\mathbf{i}}$. Note that the value of the last occurence of $\mathbf{i}$ in *every* $\mathbf{P}$ is $\mathbf{n}-1$. Thus, as long as $\mathbf{n} \geq 2$, $\mathbf{i}$ is a sequence of integers in range of 1 to $(\mathbf{n}-1)$.

Given we had to answer the question of *What's the chances of rolling a six in 100 rolls of a dice?* which translates to $\mathbf{n} = 100$, then $\mathbf{P}$ would probably fill more than one page. Thanks to the summation of an infinite sequence operator called $\mathbf{\Sigma}$ *(sigma)* we can save us the trouble and instead write ...

$$\mathbf{P} = \frac{1}{6} + \sum_{i=1}^{n-1} \frac{5}{6}^{i} \times \frac{1}{6}$$

... which is the answer of the original question.